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Quantization and scattering in the IIB $\mathrm{SL}(2,\mathbb{Z})$ covariant superstring

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Abstract: We rewrite the $\mathrm{SL}(2,\mathbb{Z})$ covariant worldsheet action for the IIB string proposed by Townsend in a Polyakov form. In a flat background the formalism yields separate (p,q) sectors. In each one the action is that of the IIB string action with the string slope parameter α' replaced by its $\mathrm{SL}(2,\mathbb{Z})$ analogue α'_{pq}. $\mathrm{SL}(2,\mathbb{Z})$ invariant graviton scattering amplitudes are obtained from those of the fundamental $(1,0)$ string by summing over the different sectors. The tree-level four-graviton amplitude in this formalism differs from a previously conjectured non-perturbative form; both yield the same expansion in order α'^3.

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1. Introduction

Exploring the non-perturbative structure of the IIB superstring is relevant for an understanding of its dynamics and for a potential formulation of the conjectured M-theory. Progress in the pinning down of the derivative expansion of the IIB effective action has been made in the recent past, especially with regards to the conjectured self-duality of the IIB theory. Relevant to this program is an attempt to reorganize the perturbative expansion in a manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant fashion and to find further contributions or structure in the derivative expansion.

The conjectured $\mathrm{SL}(2,\mathbb{Z})$ S-duality of IIB string theory maps the fundamental string into a solitonic D-string. This led Townsend to directly incorporate the self-duality at the string level in an $\mathrm{SL}(2,\mathbb{Z})$ covariant worldsheet action for the bosonic fields of the superstring [1]. A subsequent supersymmetric version was constructed by him together with Cederwall [2]. These actions are special in that the tension is generated dynamically by a two-dimensional worldsheet gauge field that has no local degrees of freedom. Classically these fields may be integrated out and the solution to the field equations contains an integration constant which acts as the tension. It is the presence of this extra worldsheet gauge field that allows one to covariantize the action with respect to the conjectured $\mathrm{SL}(2,\mathbb{Z})$ S-duality. One may combine it with the Born-Infeld gauge field of the D-string into an $\mathrm{SL}(2,\mathbb{Z})$ doublet and a hamiltonian analysis then shows that the tension of this self-dual string is promoted to its $\mathrm{SL}(2,\mathbb{Z})$ analogue [1, 3, 4]

$$T_{pq} = \frac{1}{2\pi\alpha'_{pq}} = \frac{1}{2\pi\alpha'}\sqrt{q^2 e^{-2\phi} + (p+q\chi)^2}\,. \tag{1.1}$$

Here ϕ and χ denote the vacuum expectation values of the dilaton and axion respectively (the string coupling is $g_s = e^{\phi}$).

In a separate development Russo has made an ansatz for the non-perturbative four-graviton amplitude of the ten-dimensional IIB superstring theory [5, 6]. An origin of this ansatz lays in an SL$(2,\mathbb{Z})$ invariant set of corrections to the IIB low energy effective action at order R^4 in derivatives [7]–[13]. As the non-perturbative action of S-duality mixes the weak and strong coupling regimes of the string, these results generate full non-perturbative answers for the low energy process of graviton scattering. An interesting feature of Russo's ansatz is that it may be obtained from the standard four-graviton amplitude by replacing the tension with its SL$(2,\mathbb{Z})$ covariant analogue T_{pq} of equation (1.1) provided one incorporates the different (p,q) sectors in an appropriate way. This has led to the speculation that this amplitude may be computable from a SL$(2,\mathbb{Z})$ covariant worldsheet action.

In this letter we discuss quantization of this worldsheet action. By the appropriate introduction of auxiliary fields we may linearize the Schild form of the SL$(2,\mathbb{Z})$ covariant action to a Polyakov form, from which we know how to compute explicit amplitudes for the fundamental string. We will see that in the SL$(2,\mathbb{Z})$ covariant Polyakov formulation the field strengths of the required SL$(2,\mathbb{Z})$ worldsheet gauge fields appear linearly as they do in the manifestly four-dimensional formulation in [14]. New is the presence of a Lagrange multiplier field which forces the whole lagrangean to vanish. The Lagrange multiplier times linear field strength term is familiar from lagrangean formulations of T-duality [15, 16, 17]. In particular gauge invariance forces the constant zero mode of the Lagrange multiplier field to be quantized [17, 18, 19]. Furthermore the constraint implied by the linear appearance of the gauge field requires the non-zero modes to vanish. As this Lagrange multiplier appears as an overall term in front of the whole lagrangean its constant part in effect becomes the SL$(2,\mathbb{Z})$ covariant tension T_{pq}. The remaining action is that of the free string. We thus establish that amplitudes in the (p,q) sector are given by those of the fundamental string after substitution with T_{pq}, as is required by S-duality.

For the SL$(2,\mathbb{Z})$ covariant superstring the analysis is identical and yields the Green-Schwarz string with tension T_{pq}, but as usual kappa-symmetry prevents covariant quantization. The believed equivalence between the GS string and the RNS formalism allows us to posit the use of the latter instead.

The tension arises dynamically in the SL$(2,\mathbb{Z})$ covariant formulation and each of the (p,q) strings is a groundstate of the worldsheet theory. The worldsheet path integral then requires the sum over each of these different sectors. The use of the naive perturbative string vertex operators in each sector yields SL$(2,\mathbb{Z})$ covariant amplitudes as the sum over the individual ones from each (p,q) string. The amplitude for four-graviton scattering derived in this way reproduces the complete R^4 term in the IIB low energy effective action as does the conjecture of Russo. In summing over the amplitudes of the (p,q) strings one counts the massless graviton tree exchange an infinite number of times, but this has no effect on the R^4 higher derivative term in the effective action; that the infinite summation over the (p,q) strings gives rise to the R^4 term was noticed in [12]. The conjectured form of Russo on the other hand resembles the product of the amplitudes over the different sectors and contains just a single massless pole. A possible modification of the vertex operators with factors depending on the worldsheet gauge fields is unlikely to change the sum over the (p,q) sectors into a product. Aside from the number of massless poles, which amplitude yields the correct non-perturbative information for the IIB string can only be determined at the higher-derivative level.

The amplitude derived here is a tree-level result in the worldsheet genus expansion of the $\mathrm{SL}(2,\mathbb{Z})$ covariant string. Russo has conjectured that the product form should receive $\mathrm{SL}(2,\mathbb{Z})$ covariant corrections as well. In the complete form of the non-perturbative IIB graviton scattering amplitude one would expect cuts required by unitarity as well as imaginary parts arising from poles off the real axis due to the unstable states of the (p,q) strings. The problem for both $\mathrm{SL}(2,\mathbb{Z})$ covariant amplitudes is identifying the correct small parameter which governs the perturbative expansion of the $\mathrm{SL}(2,\mathbb{Z})$ covariant string; for each (p,q) string the natural expansion parameter is the (p,q) dilaton, which measures the topology of the Riemann surface. There does exist a proposal by Berkovits for a manifestly S-dual worldsheet action for the IIB string on a Calabi-Yau three-fold which contains a coupling to the worldsheet curvature [14].

The organization of this work is as follows. First we derive a Polyakov form of the manifestly $\mathrm{SL}(2,\mathbb{Z})$ covariant string. In section 3 we solve the constraints arising from the field equations and show that the resulting action is that of separate (p,q) strings. The IIB superstring is presented in section 4. In section 5 we generate amplitudes for the $\mathrm{SL}(2,\mathbb{Z})$ covariant superstring and analyze the results.

2. Polyakov action for the $\mathrm{SL}(2,\mathbb{Z})$ covariant IIB string

We first briefly review the origins of the action that lends itself to an $\mathrm{SL}(2,\mathbb{Z})$ covariant formulation [1, 20]. This formulation is in terms of an action where the tension is generated as a dynamical variable [1, 20],

$$S_T = \int d^2\sigma\ \frac{\lambda^2}{2}\left[\frac{\det(G_{\alpha\beta})}{(2\pi\alpha')^2} + (\epsilon^{\alpha\beta}F_{\alpha\beta})^2\right]. \tag{2.1}$$

Here $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol in two dimensions. The field $\lambda(\sigma)$ transforms under worldsheet coordinate transformations as $\lambda \to \lambda' = \lambda|\partial\sigma'/\partial\sigma|^{-1/2}$; $G_{\alpha\beta}$ is the pullback of the space-time metric $G_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ and $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]}$ a worldsheet field strength. The equation of motion for A_γ reads $\partial_\gamma(\lambda^2\epsilon^{\alpha\beta}F_{\alpha\beta}) = 0$. After substituting the general solution $\epsilon^{\alpha\beta}F_{\alpha\beta} = T/\lambda^2$ for constant tension T back into the action one recovers the Schild action. This reduces to the Nambu-Goto form after one integrates out the remaining auxiliary field λ. The action (2.1) is in addition invariant under the rigid spacetime scale variations

$$X \to X' = e^t X\,, \qquad \lambda \to \lambda' = e^{-2t}\lambda\,, \qquad A_\alpha \to A'_\alpha = e^{2t}A_\alpha\,. \tag{2.2}$$

This invariance was a motivation behind its construction [20].

The appearance of the extra worldsheet gauge field A_γ makes the above action a starting point for a formulation of the $\mathrm{SL}(2,\mathbb{Z})$ covariant IIB string. A comparison of the regular string action,

$$S_F = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{|G_{\alpha\beta}|} + \frac{1}{2\pi\alpha'}\int d^2\sigma\ \epsilon^{\alpha\beta}B^{NS}_{\alpha\beta}\,, \tag{2.3}$$

with the Born-Infeld action of the D-string,

$$S_D = T_{D1}\int d^2\sigma \sqrt{|G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} + B_{\alpha\beta}|} + T_{D1}\int d^2\sigma\ \epsilon^{\alpha\beta}B^{RR}_{\alpha\beta}\,, \tag{2.4}$$

reveals that the absence of a gauge field on the worldvolume of the fundamental string is a fundamental difference between the two. If one disregards considerations of the D-string action as a solely "effective one" and implements S-duality at this level there should be a way to also introduce a worldsheet gauge field for the fundamental string, though without changing its dynamics. This is what the action (2.1) accomplishes. The similarity between the D-string action and the action (2.1) for the fundamental string with dynamical tension can be made even more suggestive by realizing [20] that the action (2.1) is equivalent to the Born-Infeld form

$$S_T = \int d^2\sigma\ \frac{\lambda^2}{2(2\pi\alpha')^2}\det(G_{\alpha\beta} + 2\pi\alpha'\tilde{F}_{\alpha\beta})\,. \tag{2.5}$$

The tilde over the field-strength is added to distinguish this worldsheet gauge field which generates the tension from the Born-Infeld gauge field of the D-string.

To construct the $\mathrm{SL}(2,\mathbb{Z})$ covariant action [2] one thus introduces two worldsheet gauge fields $A^{(i)}_\alpha$ and combines them into a doublet. Moreover, the field strengths are now both extended with the doublet of pullbacks of the NS-NS and R-R anti-symmetric tensor fields

$$F^{(i)}_{\alpha\beta} = \partial_{[\alpha}A^{(i)}_{\beta]} + \frac{B^{(i)}_{\alpha\beta}}{2\pi\alpha'}\,, \tag{2.6}$$

to the well-known $F + B$ combination.

To complete the construction one also needs the background scalars of the theory: the dilaton and the axion. These belong to the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \simeq \mathrm{SU}(1,1)/\,\mathrm{U}(1)$ and are denoted by the doublet $\mathcal{Y}_i$. They combine with the $F^{(i)}_{\alpha\beta}$ into an $\mathrm{SU}(1,1)$ invariant field strength

$$\mathcal{F}_{\alpha\beta} = \mathcal{Y}_i F^{(i)}_{\alpha\beta}\,. \tag{2.7}$$

To relate this to a more familiar description of the background scalars one may make the U(1) gauge choice $\mathcal{Y}_1 = e^{\phi/2}$ and $\mathcal{Y}_2 = e^{\phi/2}\chi + ie^{-\phi/2}$.

The $\mathrm{SL}(2,\mathbb{Z})$ covariant action is then given by,

$$S_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma \, \frac{\lambda^2}{2}\left[\frac{\det(G)}{(2\pi\alpha')^2} + \Phi\bar{\Phi}\right]. \tag{2.8}$$

Here $\det(G)$ is the determinant of the pullback of the Einstein frame metric; it is this metric which is $\mathrm{SL}(2,\mathbb{Z})$ invariant. The field Φ is the Hodge dual of the SU(1,1) invariant field strength: $\Phi = \epsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta}$.

To be able to quantize the theory and compute scattering elements we would rather work with a Polyakov form of the action. In order to rewrite the above action in such a manner, we shall make use of the following exponential integral,

$$\int_{-\infty}^{\infty} dx \; e^{-x^2-\frac{a^2}{x^2}} = \sqrt{\pi}e^{-2a}\,, \tag{2.9}$$

to change the unit power determinant $\det(G)$ to the square root $\sqrt{\det(G)}$ and obtain a Nambu-Goto form of the string action. The above integral illustrates the formal equivalence of the Schild action with the Nambu-Goto one. Eq. (2.9) is the reason for using a square for the worldsheet scalar field. An analogous functional form of (2.9) is then also exact in the path integral.

The first step is to obtain a term in the action that behaves as $1/\lambda^2$. This is straightforward as the form of the gauge field couplings in (2.8) may be reproduced by the introduction of an auxiliary source field k coupling to Φ, whose kinetic term will then exhibit the desired behaviour.

$$S_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma \, \frac{\lambda^2}{2(2\pi\alpha')^2}\det(G) - \frac{1}{2\lambda^2}k\bar{k} + \frac{1}{2}\left(\bar{k}\Phi + \bar{\Phi}k\right)\,. \tag{2.10}$$

Integrating out k gives back the previous $\mathrm{SL}(2,\mathbb{Z})$ covariant action.

We next integrate out the scalar field λ using the identity in eq. (2.9). This yields

$$S_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma \, -\frac{1}{4\pi\alpha'}\sqrt{-\det(G)}\sqrt{k\bar{k}} + \frac{1}{2}\left(\bar{k}\Phi + \bar{\Phi}k\right)\,. \tag{2.11}$$

Redefining $k = \rho e^{i\theta}$, we obtain the Nambu-Goto form of the IIB $\mathrm{SL}(2,\mathbb{Z})$ covariant action

$$S_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma \, -\frac{\rho}{4\pi\alpha'}\sqrt{-\det(G)} + \frac{1}{2}\left(\rho e^{-i\theta}\Phi + \rho\bar{\Phi}e^{i\theta}\right)\,. \tag{2.12}$$

Rescaling the field $\rho \to (2\pi\alpha')\rho$ does not eliminate its effect in the action because the Φ dependent terms will pick up this factor. (We ignore the jacobian effects due to the field redefinitions since we are examining classical actions.)

It is now straightforward to introduce an auxiliary worldsheet metric $g_{\alpha\beta}$ and obtain the Polyakov form of the action. The scalar field ρ enters multiplicatively in (2.12) and therefore does not alter the form of the worldsheet metric field equations. The gauge field terms involving Φ are also worldsheet coordinate invariant provided the epsilon used to contract the field strength is the density and not the tensor

$$\Phi = \epsilon^{\alpha\beta}_{\text{Levi-Civita}}\mathcal{F}_{\alpha\beta} = \sqrt{g}\varepsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta}\,. \tag{2.13}$$

The Polyakov form of the action is thus

$$S^{Pol}_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma\ \frac{\rho}{4\pi\alpha'}\sqrt{g}g^{\alpha\beta}G(X)_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\int d^2\sigma\,\left[\rho e^{-i\theta}\Phi + \rho\bar{\Phi}e^{i\theta}\right]\,, \tag{2.14}$$

and is worldsheet diffeomorphism and Weyl invariant.

3. IIB (p,q) strings

To analyze the theory we choose a (conformally) flat background in the Einstein frame $G^E_{\mu\nu} = \eta^E_{\mu\nu} \equiv e^{-\phi/2}\eta^{String}_{\mu\nu}$, $B^{(i)}_{\mu\nu} = 0$ and a constant dilaton and axion ϕ and χ:

$$\begin{aligned} S^{Pol}_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma\ \frac{\rho}{4\pi\alpha'}\ \sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - \rho\cos(\theta)e^{\phi/2}\epsilon^{\alpha\beta}F^{(1)}_{\alpha\beta} - \\ - \left[\rho\cos(\theta)e^{\phi/2}\chi - \rho\sin(\theta)e^{-\phi/2}\right]\epsilon^{\alpha\beta}F^{(2)}_{\alpha\beta}\,. \end{aligned} \tag{3.1}$$

The preceding steps have linearized the Nambu-Goto form of the action to the more familiar Polyakov one from which we know in principle how to compute S-matrix elements of the string fields (e.g. massless fields in the gravitational multiplet). Note that naively integrating out the world-sheet gauge fields forces the Lagrange multiplier field ρ to be constant: it yields the tension of the string, just as in the action (2.1). We now make this last step more precise and find that the tension is in fact the (p,q) string tension T_{pq} of equation (1.1).

The above form of the action is similar to that encountered in the path integral formulation of T-duality [15, 16, 17]. In the latter one gauges the spacetime isometry of one of the coordinates, $\delta X_T = \epsilon$, by the introduction of a worldsheet gaugefield. One also adds a Lagrange multiplier whose field equation enforces the connection to be flat. In other words the gauge field is pure gauge and there are no new local degrees of freedom. After choosing the gauge where the potential vanishes one recovers the original action; integrating out the gauge field on the other hand yields the T-dual action.

It is this Lagrange multiplier term that we see recurring twice in the last two terms in (3.1). For each such term[1]

$$\int \ell_i F^{(i)}\,, \qquad i = 1,2\,, \qquad \text{with } F^{(i)} = \epsilon^{\alpha\beta}F^{(i)}_{\alpha\beta} \tag{3.2}$$

[1] A boundary term is required to make this and the following steps exact. This term has been suppressed in all of the above. [1, 21]

we can split the scalar Lagrange multiplier field into a constant zero-mode and non-constant piece (see e.g. [19])

$$\ell_i = \ell_i^{(0)} + \tilde{\ell}_i \,, \qquad \partial_\alpha \ell_i^{(0)} = 0 \,; \qquad \tilde{\ell}_i \Big|_{\sigma,\tau=0} = 0 \,. \tag{3.3}$$

On a non-trivial world-sheet single-valuedness of the path-integral under large gauge transformations forces the constant part multiplying the field strength $\ell_i^{(0)}$ to be an integer up to an appropriate normalization

$$\int_{T_2} F^{(i)} \in \mathbb{Z} \;\rightarrow\; \ell_i^{(0)} \in \mathbb{Z} \,. \tag{3.4}$$

This quantization follows from the gauge group being U(1) rather than $\mathbb{R}$ [18, 19].

Integration by parts, however, removes the quantized constant part $\ell^{(0)}$ from the terms (3.2)

$$\int \ell_i F_i = -\int \epsilon^{\alpha\beta} \partial_\alpha \ell_i A^i_\beta \equiv -\int \epsilon^{\alpha\beta} \partial_\alpha \tilde{\ell}_i A^i_\beta \,. \tag{3.5}$$

Now integrating over the gauge field as a Lagrange multiplier forces the remaining part $\tilde{\ell}_i$ to vanish. This means that the full coefficient fields ℓ_i are constant and quantized; the gauge fields have been accounted for as Lagrange multipliers whose constraint has been explicitly enforced.

In the context of the action in (3.1) the quantized constant parts are [17, 18]

$$\begin{aligned} \ell_1^{(0)} &= \rho_0 \cos(\theta_0) e^{\phi/2} = q \,; \\ \ell_2^{(0)} &= \rho_0 \cos(\theta_0) e^{\phi/2} \chi - \rho_0 \sin(\theta_0) e^{-\phi/2} = p \,, \end{aligned} \tag{3.6}$$

or

$$\begin{aligned} \rho_0 \sin(\theta_0) &= p e^{\phi/2} + q \chi e^{\phi/2} \,, \\ \rho_0 \cos(\theta_0) &= q e^{-\phi/2} \,. \end{aligned} \tag{3.7}$$

In other words the constant part of ρ equals

$$\rho_0 = e^{\phi/2} \sqrt{(p + q\chi)^2 + q^2 e^{-2\phi}} = 2\pi\alpha' e^{\phi/2} T_{pq} \,, \tag{3.8}$$

and contains precisely the form of the SL$(2,\mathbb{Z})$ covariantized tension in eq. (1.1). Note that as the gauge field F can be extended to the open string combination $F+B$ the integers p and q are indeed the charges under the NS-NS and R-R antisymmetric tensor field. The extra overall factor of $e^{\phi/2}$ is what is required to transform the Einstein metric to the string frame. The resulting theory is just that of the regular string but with the tension T replaced by its (p,q) quantized value T_{pq}.

4. IIB $\mathrm{SL}(2,\mathbb{Z})$ covariant superstring

So far we have discussed an approach to the $\mathrm{SL}(2,\mathbb{Z})$ covariant IIB superstring involving the bosonic worldsheet degrees of freedom only. Our focus, however, is the superstring as it is the latter which is expected to possess the $\mathrm{SL}(2,\mathbb{Z})$ S-duality. Formally the $\mathrm{SL}(2,\mathbb{Z})$ covariant action for the IIB superstring is identical to the action (2.8) containing only the bosonic fields, except that one makes all fields *spacetime* supersymmetric [2]. The action is thus again

$$S_{\mathrm{SL}(2,\mathbb{Z})} = \int d^2\sigma \; \frac{\lambda^2}{2}\left[\frac{\det(G_{\alpha\beta})}{(2\pi\alpha')^2} + \Phi\bar{\Phi}\right], \tag{4.1}$$

with

$$\Phi \equiv \epsilon^{\alpha\beta}\mathcal{Y}_i\left(F^{(i)}_{\alpha\beta} + \frac{B^{(i)}_{\alpha\beta}}{2\pi\alpha'}\right), \tag{4.2}$$

but the pullback of the metric and the antisymmetric tensor fields are now defined as

$$\begin{aligned} G_{\alpha\beta} &= \eta_{ab}E^a_M E^b_N \partial_\alpha Z^M \partial_\beta Z^N\,, \\ B^{(i)}_{\alpha\beta} &= \partial_\alpha Z^M \partial_\beta Z^N B^{(i)}_{MN}\,. \end{aligned} \tag{4.3}$$

Here $Z^M = (X^\mu, \theta_1^{\mathbf{a}}, \theta_2^{\mathbf{a}})$ and $E^A_M = (E^a_M, E^{\mathbf{a_1}}_M, E^{\mathbf{a_2}}_M)$ are the embeddings of the worldsheet into the coordinates and the vielbeins for the target space of the IIB superstring. Supersymmetrizing does not change the world-sheet Lagrange multipliers ρ, θ or the world-sheet gauge fields $F^{(i)}$ [2]. The steps in section 3 then carry over unaltered to the supersymmetric string. In particular, the Polyakov action

$$\begin{aligned} S^{Super}_{\mathrm{SL}(2,\mathbb{Z})} &= \int d^2\sigma \; \frac{\rho}{4\pi\alpha'}\sqrt{g}g^{\alpha\beta}\partial_\alpha Z^M \partial_\beta Z^N E^a_M E_{Na} - \frac{\rho}{2}e^{-i\theta}\Phi - \frac{\rho}{2}e^{i\theta}\bar{\Phi} \\ &= \int d^2\sigma \; \frac{\rho}{4\pi\alpha'}\sqrt{g}g^{\alpha\beta}\partial_\alpha Z^M \partial_\beta Z^N E^a_M E_{Na} - \\ &\qquad - \frac{\rho}{2}e^{-i\theta}\epsilon^{\alpha\beta}\mathcal{Y}_i\left(F^{(i)}_{\alpha\beta} + \partial_\alpha Z^M \partial_\beta Z^N \frac{B^{(i)}_{MN}}{2\pi\alpha'}\right) - \\ &\qquad - \frac{\rho}{2}e^{i\theta}\epsilon^{\alpha\beta}\bar{\mathcal{Y}}_i\left(F^{(i)}_{\alpha\beta} + \partial_\alpha Z^M \partial_\beta Z^N \frac{B^{(i)}_{MN}}{2\pi\alpha'}\right), \end{aligned} \tag{4.4}$$

reduces to that of the Green-Schwarz string with again the tension T replaced by that of the (p,q) string. Indeed, as was noted in [2] the terms involving the superfield B_{MN} containing the antisymmetric tensor fields yield the Wess-Zumino-Witten term in flat superspace [22, 23]. Note also the presence of the D-string $\int \chi(F+B)^{(2)}$ coupling [24].

This action is almost identical to the one proposed by Berkovits for the manifestly $\mathrm{SL}(2,\mathbb{Z})$ covariant IIB string compactified on a Calabi-Yau three-fold [14]. The difference is roughly that in our case the $\mathrm{SL}(2,\mathbb{Z})$ inert Lagrange multiplier ρ also multiplies the term involving the supervielbeins.

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5. $\mathrm{SL}(2,\mathbb{Z})$ amplitudes

We now turn to the computation of $\mathrm{SL}(2,\mathbb{Z})$ invariant scattering amplitudes. The motivation for working with an $\mathrm{SL}(2,\mathbb{Z})$ invariant formulation is to obtain $\mathrm{SL}(2,\mathbb{Z})$ invariant (and dual) scattering expressions directly. However, a problem arises in that at present we do not yet know how to covariantly quantize the ten-dimensional kappa-symmetric GS-string. Following the usual light-cone arguments that equate the GS-string with the RNS spinning string we may replace the tension T with that of the (p,q) string there as well; alternatively one may quantize the GS-string on the light-cone directly. In the RNS form we can then use the standard techniques of string perturbation theory.

A (p,q) string "vacuum" arises from solving the field equations for the $\mathrm{SL}(2,\mathbb{Z})$ fields ρ and $F^{(i)}$ and spontaneously breaks the $\mathrm{SL}(2,\mathbb{Z})$ covariance of the action: each of the possible values for $\rho_0^{(p,q)}$ is an equally legitimate solution to the field equations. In quantizing the string action the functional integral over ρ is then equivalent to a sum over the contributions of each of the (p,q) strings

$$
\begin{aligned}
Z^{\mathrm{SL}(2,\mathbb{Z})} &= \int \frac{[dX^\mu][dg]}{\mathcal{N}}[d\rho][dA^{(i)}]\; e^{-S_{\mathrm{SL}(2,\mathbb{Z})}} \\
&= \sum_{(p,q)'} \int \frac{[dX^\mu][dg]}{\mathcal{N}}\; e^{-T_{pq}S_0}\,,
\end{aligned}
\tag{5.1}
$$

where the remaining integration is over the matter and gravitional superfields on the world-sheet; $\mathcal{N}$ represents the modding out of the local worldsheet symmetries of the string. As the notation indicates we have limited the sum to relatively prime pairs. If (p,q) were not relatively prime then the tension T_{pq} would be the multiple of that of a $T_{p'q'}$ string with coprime charges; as this configuration is a superposition of independent (p,q) strings [3] we consider this as overcounting in the functional integration. In addition, the charges (p,q) should not be simultaneously zero as the action vanishes in this case.

The partition function thus remains invariant under an $\mathrm{SL}(2,\mathbb{Z})$ transformation though perturbative amplitudes in each individual (p,q) sector are not. $\mathrm{SL}(2,\mathbb{Z})$ covariant amplitudes are obtained from the full partition function (5.1) after summing over the (p,q) sectors,

$$
A_n^{\mathrm{SL}(2,\mathbb{Z})} = \sum_{(p,q)'} \int \frac{[dX_m][dg]}{\mathcal{N}_n}\; e^{-T_{pq}S_0}\; V_1 V_2 \cdots V_n\,.
\tag{5.2}
$$

For $\mathrm{SL}(2,\mathbb{Z})$ invariant vertex operators the summation over ρ manifestly leads to $\mathrm{SL}(2,\mathbb{Z})$ invariant results through perturbation theory in the covariant string: the amplitudes are the ($\mathrm{SL}(2,\mathbb{Z})$ invariant) sum of (p,q) sub-amplitudes. As the (p,q) string only differs in the tension from the fundamental string the naive choice is to use

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the regular perturbative string vertex operators in each sector. Amplitudes for the (p,q) string are then obtained from those of the fundamental string by substituting $T = 1/2\pi\alpha'$ with T_{pq} in accordance with S-duality. For graviton scattering we use in each $(p,q)'$ sector the standard vertex operator for the emission of a gravitational state.

One might argue that the use of the naive vertex operators is incorrect because of the extra fields in the $\mathrm{SL}(2,\mathbb{Z})$ covariant world-sheet action. For example in (4.4) the coupling to the background Einstein metric includes a factor of the Lagrange multiplier field ρ. On-shell this is constant but not (p,q) independent and amounts to a normalization condition depending on (p,q). The vertex operator should in principle be compatible with $\mathrm{SL}(2,\mathbb{Z})$ invariance and we shall take the (p,q) normalization to be unity.

The four-graviton amplitude found from (5.2) is the sum over the (p,q) four-graviton amplitudes,

$$A_4^{\mathrm{SL}(2,\mathbb{Z})} = \sum_{(p,q)'} e^{-2\phi}\kappa_{pq}^2 E \frac{(\alpha'_{pq})^3}{\bar{s}\bar{t}\bar{u}} \frac{\Gamma(1-\bar{s})\Gamma(1-\bar{t})\Gamma(1-\bar{u})}{\Gamma(1+\bar{s})\Gamma(1+\bar{t})\Gamma(1+\bar{u})}\,, \tag{5.3}$$

with the barred invariants given by,

$$\bar{s} = \frac{\alpha'_{pq}}{4} s\,, \qquad s+t+u=0\,. \tag{5.4}$$

Here κ_{pq} is the gravitational coupling constant, including a (p,q) dependent normalization and E is the kinematic factor (e.g. [26]) depending on the polarizations (we shall normalize the p,q dependence in $\kappa_{p,q}$ to unity). After transforming to the Einstein frame, where $s = \sqrt{\tau_2} s_E$, the $\mathrm{SL}(2,\mathbb{Z})$ invariance is seen explicitly.

Russo [6] has analyzed why this $\mathrm{SL}(2,\mathbb{Z})$ invariant amplitude in (5.3) does not yield complete non-perturbative results for string amplitudes. The form in (5.3) counts the pole of the gravitational exchange (i.e. massless) an infinite number of times, once from each of the (p,q) sectors. Also, when expanded around weak coupling, the expression contains odd powers of g_s which cannot occur in closed string perturbation theory. This is in contrast to the ansatz in [5],

$$A_4 = e^{-2\phi}\kappa^2 E \frac{1}{stu} \prod_{(p,q)'} \frac{\Gamma(1-\bar{s})\Gamma(1-\bar{t})\Gamma(1-\bar{u})}{\Gamma(1+\bar{s})\Gamma(1+\bar{t})\Gamma(1+\bar{u})}\,, \tag{5.5}$$

which is also given in string frame. Instead of a summation over the various sectors it takes the form of a product in such a way that the expression contains a single massless pole, corresponding to the exchange of a single gravitational mode.[2] It is unclear how a further dressing with the $\mathrm{SL}(2,\mathbb{Z})$ peculiar fields could change the sum in (5.3) into a product.

[2]The product is reminiscent of the closed/open string amplitude relations at tree-level: $A_4^{closed} = \sum A_4^{open}\bar{A}_4^{open}$. However, because the graviton is spin-two it can not arise from an infinite product of non-zero spin states.

The massless excitations of the different (p, q) sectors are not independent but should be identified under the duality. The current formalism does not contain this prescription, but an analysis of the above action similar to that of the D-string in [25] could provide this. The IIB supersymmetry algebra of the D-string action contains central charges, which means that the massless excitations — the gravitational multiplet — are projected out [25]. A projector operator, similar to the GSO, may be inserted by hand to eliminate the overcounting associated with the massless tree exchange (although we do not have a world-sheet construction of this).

After subtraction of the massless poles, both expressions (5.3) and (5.5) give rise to the same R^4 term in the effective action. Our result in (5.3) bears out the interpretation of this term given in [12]. Aside from the indirect arguments of Russo regarding the massless poles and the powers of g_s we cannot in this order tell which of the two results yields the correct non-perturbative string scattering amplitudes. A comparison could be made at the next order in the derivative expansion or for higher point amplitudes. The formalism presented here allows one to compute these using the standard rules of string perturbation theory.

It would be interesting to find a prescription for the computation of the scattering amplitude that leads to the conjecture of Russo. For example, instead of summing over the different sectors of the solution space of ρ, i.e. $\rho_0^{(p,q)}$, we could integrate naively with measure factor $\prod d\rho_0^{(p,q)}$ in the spirit of the heuristic proposal in [6]. A placement of the product within the integration is then crucial. Using the form

$$\int d\mu_4 \left[\prod_{(p,q)'} d\rho_0^{(p,q)} \right] e^{-T_{(p,q)} S} \prod_{i=1}^{4} V_i(z, \bar{z}) \tag{5.6}$$

gives an infinite tension in the exponential; this occurs either including or not including the vertex operators in the product (in the latter case one would have to normalize by dividing by the infinite helicity product $\prod_{(p,q)'} E/\bar{s}\bar{t}\bar{u}$, which does not seem sensible.).

Though we have not analyzed the analytic structure of (5.3) in detail, it appears that both expressions (5.3) and (5.5) have the fundamental problem that they do not contain any cuts or poles off the real axis. At weak string coupling all the excited states of the $(p, q) \neq (1, 0)$ strings are expected to be unstable and the full non-perturbative four-graviton amplitude should reflect this. To this extent neither expression can be the full answer but only a first approximation. The natural extension of the results here would be to consider amplitudes on higher genus surfaces as well; the amplitude (5.3) arose at string tree-level, i.e. superspherical worldsheet. Similarly, the product form is conjectured to be a tree-level result [6]. The higher-genus corrections could generate the required cuts.

This poses the fundamental question as to what parameter governs the genus expansion in this formulation. The correct dilaton coupling in the GS-string is not yet known. In the RNS formalism the dual (p, q) dilaton measures the expansion in each (p, q) sector and taking $\phi_{p,q} = \phi$ leads to explicit $\mathrm{SL}(2, \mathbb{Z})$ invariant results. The expression (5.3) implictly takes each of the (p, q) coupling constants, $e^{-\phi}$, simultaneously infinitesimally small. The amplitude one generates this way is manifestly S-dual, though the worldsheet genus expansion of the $\mathrm{SL}(2, \mathbb{Z})$ string no longer corresponds to a perturbative expansion in g_s. It is not clear whether there exists an extension of the dilaton that is able to govern the world sheet expansion of the $\mathrm{SL}(2, \mathbb{Z})$ covariant string in an invariant way. This problem has been solved by the introduction of additional fields for the manifestly four dimensional formulation of the IIB $\mathrm{SL}(2, \mathbb{Z})$ covariant superstring by Berkovits [14] but a corresponding formalism is as yet unknown for the ten-dimensional case.

6. Conclusion

In summary we have shown that the $\mathrm{SL}(2, \mathbb{Z})$ covariant action for the IIB string of [1, 2] may be rewritten in a Polyakov form. This is a first step on the way to quantization and manifestly organizes the perturbative expansion in an $\mathrm{SL}(2, \mathbb{Z})$ invariant fashion. An analysis of the worldsheet field equations shows that the groundstates of the worldsheet theory correspond to individual (p, q) strings. One is required to sum over the various sectors in the first quantized partition function, with each term corresponding to the contribution for the individual (p, q) superstring. $\mathrm{SL}(2, \mathbb{Z})$ covariant four- and higher point amplitudes can be computed following this approach by taking the sum over the individual (p, q) ones. We find the four-graviton scattering amplitude; it reproduces the modular $f(\tau, \bar{\tau})$ coefficient of the R^4 term in the low energy effective action of [8]. The conjectured $\mathrm{SL}(2, \mathbb{Z})$ covariant amplitude of Russo does so as well but the latter contains a single massless pole, which is as one expects, whereas ours counts an infinite number before projection, one from each sector. These massless poles ought to be identified under the action of $\mathrm{SL}(2, \mathbb{Z})$. A problem is the identification of the appropriate small parameter that measures the world-sheet genus expansion in an $\mathrm{SL}(2, \mathbb{Z})$ invariant way. This should be an extension of the (p, q) dilaton which governs the expansion for each (p, q) sector.

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